

# The effects of a discontinues weight for a problem with a critical nonlinearity

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## Abstract

We study the minimizing problem  $\inf \left\{ \int_{\Omega} p(x) |\nabla u|^2 dx, u \in H_0^1(\Omega), \|u\|_{L^{\frac{2N}{N-2}}(\Omega)} = 1 \right\}$  where  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^N$ ,  $N \geq 3$  and  $p$  a positive discontinuous function. We prove the existence of a minimizer under some assumptions.

Keywords : Critical Sobolev exponent, Lack of compactness, Best Sobolev constant.

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## 1 Introduction

We consider the minimizing problem

$$S(p) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} p(x) |\nabla u|^2 dx}{\left( \int_{\Omega} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}, \quad (1)$$

where  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^N$ ,  $N \geq 3$ ,  $2^* = \frac{2N}{N-2}$  and  $p$  a discontinuous function.

This problem is closely related to the best constant in Sobolev inequality in  $\mathbb{R}^N$ . It posses many interesting properties, see Talenti [14], and arising in many areas of mathematics and in a geometric context namely for example in the Yamabe problem and the prescribed scalar curvature problem see Aubin [1]. It's invariance under dilations produces a lack of compactness.

The phenomenon of lack of compactness and the failure of the Palais Smale condition of this type of problem has been the subject of several studies and it was analyzed in minute detail by Struwe [13].

In the case where  $p$  is a constant, it is well known that (1) is not achieved for a general domain  $\Omega$ . Nevertheless, Brezis and Nirenberg showed in [6] that (1) has minimizer under

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a linear perturbation. Bahri and Coron in [4] proved that the Euler equation associated to this problem is solvable when some homology group of the domain with coefficients in  $\mathbb{Z}/2\mathbb{Z}$  is nontrivial, see also the work of Coron [7]. In the case where  $p$  is a smooth positive function, we proved that the study of problem (1) depends on the behavior of the weight  $p$  near its minima, see [9] (see also [11]).

One may ask whether the lack of compactness of the variational functional associated to (1) can be made up by the discontinuity of the weight.

In this paper, we consider the discontinuous function  $p$  defined by

$$p(x) = \begin{cases} \alpha & \text{if } x \in \Omega_1 \\ \beta & \text{if } x \in \Omega_2, \end{cases} \quad (2)$$

where  $\alpha$  and  $\beta$  are some positive constants such that  $\alpha < \beta$  and  $\Omega_1, \Omega_2$  are two non empty, disjoint domains such that  $\Omega = \Omega_1 \cup \Omega_2$  and  $\Gamma_1 = \partial\Omega_1 \cap \partial\Omega_2$  is not empty. Let us remark that the general case of discontinuous function will be treated in [10].

## 2 Statements and proofs of results

Let

$$S_{\alpha, \beta} = \inf \left\{ \alpha \int_{\mathbb{R}_+^N} |\nabla u|^2 dx + \beta \int_{\mathbb{R}_-^N} |\nabla u|^2 dx, \quad u \in H^1(\mathbb{R}^N), \quad u \neq 0 \text{ in } \mathbb{R}_\pm^N, \quad \|u\|_{L^{2^*}(\mathbb{R}^N)} = 1 \right\}.$$

Set

$$S^+ = \inf \left\{ \int_{\mathbb{R}_+^N} |\nabla u|^2 dx, \quad u \in H^1(\mathbb{R}_+^N), \quad u \neq 0 \text{ in } \mathbb{R}_+^N, \quad \|u\|_{L^{2^*}(\mathbb{R}_+^N)} = 1 \right\}.$$

and

$$S^- = \inf \left\{ \int_{\mathbb{R}_-^N} |\nabla u|^2 dx, \quad u \in H^1(\mathbb{R}_-^N), \quad u \neq 0 \text{ in } \mathbb{R}_-^N, \quad \|u\|_{L^{2^*}(\mathbb{R}_-^N)} = 1 \right\}.$$

It is easy to verify that (see for example [8])

$$S^+ = S^- = \frac{S}{2^{\frac{2}{N}}}, \quad (3)$$

where  $S$  is the best constant of the Sobolev embedding defined by

$$S = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}.$$

Our main results are the following

**Theorem 1** *We have*

$$S_{\alpha, \beta} = \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S.$$

**Theorem 2**

*Let  $\Omega$ ,  $\Omega_1$ ,  $\Omega_2$  and  $p$  be as (2). Assume that the following geometrical condition (g.c.) on  $\Gamma_1$  holds: there exists an  $x_0$  in the interior of  $\Gamma_1$  such that in a neighborhood of  $x_0$ ,  $\Omega_2$  lies on one side of the tangent plane at  $x_0$  and the mean curvature with respect to the unit inner normal of  $\Omega_2$  at  $x_0$  is positive.*

*then  $S(p)$  is attained by some  $u \in H_0^1(\Omega)$ .*

**Remark 1** *Let us give simple examples for which the condition (g.c.) in Theorem 2 is fulfilled or not. Let  $\Omega = B(0, R)$ ,  $R > 1$  and  $e_1 = (1, 0, \dots, 0)$ .*

*Set  $\Omega_2 = B(e_1, R) \cap \Omega$ ,  $\Omega_1 = \Omega \setminus \overline{\Omega}_2$ ,  $\Gamma_1 = \partial\Omega_1 \cap \partial\Omega_2$  and  $x_0$  in the interior of  $\Gamma_1$ . We have condition (g.c.) holds.*

*For  $\Omega_1 = B(e_1, R) \cap \Omega$ ,  $\Omega_2 = \Omega \setminus \overline{\Omega}_1$  and  $\Gamma_1 = \partial\Omega_1 \cap \partial\Omega_2$  and  $x_0$  in the interior of  $\Gamma_1$ . We have condition (g.c.) is not satisfied. More precisely, in any neighborhood of  $x_0$ ,  $\Omega_2$  does not lie on one side of the tangent plane at  $x_0$  and the mean curvature with respect to the unit inner normal of  $\Omega_2$  at  $x_0$  is negative.*

*Let  $\Omega_1 = \{(x_1, \dots, x_N) \in \Omega \text{ s.t. } x_1 > 0\}$  and  $\Omega_2 = \{(x_1, \dots, x_N) \in \Omega \text{ s.t. } x_1 < 0\}$  and  $x_0 = 0$ . We have condition (g.c.) hold, more precisely, in any neighborhood of 0,  $\Omega_2$  lies on one side of the tangent plane at 0 but the mean curvature with respect to the unit inner normal of  $\Omega_2$  at 0 is 0.*

If  $\Gamma_1$  is flat, that is mean that mean curvature at any point of  $\Gamma_1$  is zero, then we have the following non-existence result:

**Proposition 1** *Let  $\Omega = B(0, R)$  and  $\Gamma_1 = \{x \in \Omega \mid x_N = 0\}$ . Then  $S(p)$  is not achieved.*

Indeed, If  $S(p)$  is achieved by some positive function  $u > 0$ , then there exists a Lagrange multiplier  $\mu \in \mathbb{R}$  such that  $u$  satisfies the Euler equation

$$\begin{cases} -\alpha \Delta u = \mu u^{2^*-1} & \text{in } \Omega_1, \\ -\beta \Delta u = \mu u^{2^*-1} & \text{in } \Omega_2, \\ \alpha \frac{\partial u}{\partial \nu_1} + \beta \frac{\partial u}{\partial \nu_2} = 0 & \text{On } \Gamma_1, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where  $\nu_1$  and  $\nu_2$  are respectively the outward normal of  $\Omega_1$  and  $\Omega_2$ .

On one hand we multiply (4) by  $\nabla u \cdot x$  and we integrate, on the other hand we multiply (4) by  $\frac{N-2}{2}u$  and we integrate, we obtain, after some computations, the Pohozaev identity

$$-\int_{\Gamma_1} \left[ \alpha (x \cdot \nu_1) \left| \frac{\partial u}{\partial \nu_1} \right|^2 + \beta (x \cdot \nu_2) \left| \frac{\partial u}{\partial \nu_2} \right|^2 \right] ds_x = \int_{\partial\Omega \cap \partial\Omega_1} \alpha (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^2 + \beta (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^2 ds_x,$$

where  $\nu$  is the outward of  $\partial\Omega$ . Since  $B(0, R)$  is star-shaped about 0 then  $x \cdot \nu > 0$  and then

$$-\int_{\Gamma_1} \left[ \alpha (x \cdot \nu_1) \left| \frac{\partial u}{\partial \nu_1} \right|^2 + \beta (x \cdot \nu_2) \left| \frac{\partial u}{\partial \nu_2} \right|^2 \right] ds_x > 0$$

which gives a contradiction since  $x \cdot \nu_1 = x \cdot \nu_2 = 0$  for every  $x$  in  $\Gamma_1$ . Therefore  $S(p)$  is not achieved.

**Proof of Theorem 1.** On one hand, we claim that

$$S_{\alpha, \beta} \geq \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S. \quad (5)$$

Indeed, we see that, for all  $t \in ]0, 1[$  we have

$$S_{\alpha, \beta} = \inf \left\{ \alpha \int_{\mathbb{R}_+^N} |\nabla u|^2 dx + \beta \int_{\mathbb{R}_-^N} |\nabla u|^2 dx, \quad u \in H^1(\mathbb{R}^N), \quad \|u\|_{L^{2^*}(\mathbb{R}_+^N)}^{2^*} = t, \quad \|u\|_{L^{2^*}(\mathbb{R}_-^N)}^{2^*} = 1 - t \right\}. \quad (6)$$

Therefore

$$\begin{aligned} S_{\alpha, \beta} &\geq \alpha \inf \left\{ \int_{\mathbb{R}_+^N} |\nabla u|^2 dx, \quad u \in H^1(\mathbb{R}^N), \quad \|u\|_{L^{2^*}(\mathbb{R}_+^N)}^{2^*} = t, \quad \|u\|_{L^{2^*}(\mathbb{R}_-^N)}^{2^*} = 1 - t \right\} \\ &\quad + \beta \inf \left\{ \int_{\mathbb{R}_-^N} |\nabla u|^2 dx, \quad u \in H^1(\mathbb{R}^N), \quad \|u\|_{L^{2^*}(\mathbb{R}_+^N)}^{2^*} = t, \quad \|u\|_{L^{2^*}(\mathbb{R}_-^N)}^{2^*} = 1 - t \right\} \end{aligned} \quad (7)$$

At this stage, define

$$A_t = \left\{ u \in H^1(\mathbb{R}^N), \quad \|u\|_{L^{2^*}(\mathbb{R}_+^N)}^{2^*} = t, \quad \|u\|_{L^{2^*}(\mathbb{R}_-^N)}^{2^*} = 1 - t \right\},$$

$$B_t = \left\{ u \in H^1(\mathbb{R}_+^N), \quad \|u\|_{L^{2^*}(\mathbb{R}_+^N)}^{2^*} = t \right\}$$

and

$$C_t = \left\{ u \in H^1(\mathbb{R}_-^N), \quad \|u\|_{L^{2^*}(\mathbb{R}_-^N)}^{2^*} = 1 - t \right\}.$$

We have

$$A_t \subset B_t \quad \text{and} \quad A_t \subset C_t. \quad (8)$$

We rewrite (7) as

$$S_{\alpha, \beta} \geq \alpha \inf_{A_t} \int_{\mathbb{R}_+^N} |\nabla u|^2 dx + \beta \inf_{A_t} \int_{\mathbb{R}_-^N} |\nabla u|^2 dx.$$

Using (8), we see that

$$S_{\alpha, \beta} \geq \alpha \inf_{A_t} \int_{\mathbb{R}_+^N} |\nabla u|^2 dx + \beta \inf_{A_t} \int_{\mathbb{R}_-^N} |\nabla u|^2 dx \geq \alpha \inf_{B_t} \int_{\mathbb{R}_+^N} |\nabla u|^2 dx + \beta \inf_{C_t} \int_{\mathbb{R}_-^N} |\nabla u|^2 dx. \quad (9)$$

Or, looking at (3), direct computations give that

$$\inf_{B_t} \int_{\mathbb{R}_+^N} |\nabla u|^2 dx = \frac{t^{\frac{2}{2^*}}}{2^{\frac{2}{N}}} S$$

and

$$\inf_{C_t} \int_{\mathbb{R}_-^N} |\nabla u|^2 dx = \frac{(1-t)^{\frac{2}{2^*}}}{2^{\frac{2}{N}}} S.$$

Then, (9) becomes

$$\begin{aligned} S_{\alpha, \beta} &\geq \alpha \frac{t^{\frac{2}{2^*}}}{2^{\frac{2}{N}}} S + \beta \frac{(1-t)^{\frac{2}{2^*}}}{2^{\frac{2}{N}}} S \\ &\geq \frac{1}{2^{\frac{2}{N}}} \inf_{t \in [0, 1]} \left[ \alpha t^{\frac{2}{2^*}} + \beta (1-t)^{\frac{2}{2^*}} \right] S \\ &= \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S \end{aligned}$$

which gives (5).

On the other hand, we claim that

$$S_{\alpha, \beta} \leq \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S. \quad (10)$$

In order to prove the previous claim, for every  $x \in \mathbb{R}^N$  we note  $x = (x', x_N)$  where  $x' \in \mathbb{R}^{N-1}$ .

Let  $\{u_j^+\}$  be a minimizing sequence of  $S^+$ . We define the sequence

$$u_j^-(x', x_N) = u_j^+(x', -x_N) \quad \text{for all } x \in \mathbb{R}^{N-1} \times ]+\infty, 0] \text{ and for all } j \in \mathbb{N}.$$

Easily we see that  $\{u_j^-\}$  is a minimizing sequence of  $S^-$ .

There exists  $t_0 = \frac{(\frac{\alpha}{\beta})^{\frac{N}{2}}}{1 + (\frac{\alpha}{\beta})^{\frac{N}{2}}}$  such that

$$\inf_{t \in [0, 1]} \frac{(\alpha t^{\frac{2}{2^*}} + \beta (1-t)^{\frac{2}{2^*}}) S}{2^{\frac{2}{N}}} = \frac{(\alpha t_0^{\frac{2}{2^*}} + \beta (1-t_0)^{\frac{2}{2^*}}) S}{2^{\frac{2}{N}}} = \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S.$$

We define the following functions :

$$v_j^+(x', x_N) = t_0 u_j^+(x', t_0^\theta x_N) \quad \text{for all } x \in \mathbb{R}^{N-1} \times ]0, +\infty]$$

$$v_j^-(x', x_N) = t_0 u_j^-(x', (1-t_0)^\theta x_N) \quad \text{for all } x \in \mathbb{R}^{N-1} \times ]+\infty, 0],$$

where  $\theta = \frac{2(2^* - 1)}{2^*}$ . Now, consider

$$w_j(x', x_N) = \begin{cases} v_j^+(x', x_N) & \text{for all } x' \in \mathbb{R}^{N-1} \text{ and for all } x_N \geq 0 \\ v_j^-(x', x_N) & \text{for all } x' \in \mathbb{R}^{N-1} \text{ and for all } x_N \leq 0 \end{cases} \quad (11)$$

An easy computation yields that, for large  $j$ ,  $w_j$  is a testing function for  $S_{\alpha,\beta}$  defined by (6).

Therefore

$$S_{\alpha,\beta} \leq \alpha \int_{\mathbb{R}_+^N} |\nabla v_j^+|^2 dx + \beta \int_{\mathbb{R}_-^N} |\nabla v_j^-|^2 dx.$$

Using the definitions of  $v_j^+$  and  $v_j^-$ , we obtain

$$S_{\alpha,\beta} \leq \alpha \frac{t_0^{\frac{2}{2^*}}}{2^{\frac{2}{N}}} S + \beta \frac{(1-t_0)^{\frac{2}{2^*}}}{2^{\frac{2}{N}}} S + o(1).$$

Then, using the definition of  $t_0$  and letting  $j \rightarrow +\infty$ , we obtain

$$S_{\alpha,\beta} = \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S,$$

which gives (10).

Finally, (5) and (10) give the conclusion of Theorem 1.  $\square$

The proof of Theorem 2 follows from the following two Lemmas.

**Lemma 1** *Following the hypothesis of Theorem 2, we have if  $S(p) < S_{\alpha,\beta}$  then the infimum in (1) is achieved.*

**Proof.**

We adapt the arguments of Brezis-Nirenberg ([6], proof of Lemma 2.1). Let  $\{u_j\} \subset H_0^1(\Omega)$  be a minimizing sequence for (1) that is,

$$\int_{\Omega} p(x) |\nabla u_j|^2 dx = S(p) + o(1), \quad (12)$$

$$\|u_j\|_{L^{2^*}} = 1. \quad (13)$$

Easily we see that  $\{u_j\}$  is bounded in  $H_0^1(\Omega)$  we may extract a subsequence still denoted by  $u_j$ , such that

$$u_j \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),$$

$$u_j \rightarrow u \quad \text{strongly in } L^2(\Omega),$$

$$u_j \rightarrow u \quad \text{a.e. on } \Omega,$$

with  $\|u\|_{L^{2^*}} \leq 1$ . Set  $v_j = u_j - u$ , so that

$$v_j \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega)$$

$$v_j \rightarrow 0 \quad \text{strongly in } L^2(\Omega),$$

$$v_j \rightarrow 0 \quad \text{a.e. on } \Omega.$$

Using (12) we write

$$\int_{\Omega} p(x) |\nabla u(x)|^2 dx + \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx = S(p) + o(1), \quad (14)$$

since  $v_j \rightharpoonup 0$  weakly in  $H_0^1(\Omega)$ . On the other hand, it follows from a result of [5] that

$$\|u + v_j\|_{L^{2^*}}^{2^*} = \|u\|_{L^{2^*}}^{2^*} + \|v_j\|_{L^{2^*}}^{2^*} + o(1),$$

(which holds since  $v_j$  is bounded in  $L^{2^*}$  and  $v_j \rightarrow 0$  a.e.). Thus, by (13), we have

$$1 = \|u\|_{L^{2^*}}^{2^*} + \|v_j\|_{L^{2^*}}^{2^*} + o(1) \quad (15)$$

and therefore

$$1 \leq \|u\|_{L^{2^*}}^{2^*} + \|v_j\|_{L^{2^*}}^{2^*} + o(1). \quad (16)$$

Using the definition of  $S_{\alpha,\beta}$ , extending  $v_j$  by 0 in  $\mathbb{R}^N$  (still denoted by  $v_j$ ) we obtain

$$\begin{aligned} \|v_j\|_{L^{2^*}}^2 &\leq \frac{1}{S_{\alpha,\beta}} \left[ \alpha \int_{\mathbb{R}_{+,x_0}^N} |\nabla v_j(x)|^2 dx + \beta \int_{\mathbb{R}_{-,x_0}^N} |\nabla v_j(x)|^2 dx \right] \\ &\leq \frac{1}{S_{\alpha,\beta}} \left[ \alpha \int_{\Omega \cap \mathbb{R}_{+,x_0}^N} |\nabla v_j(x)|^2 dx + \beta \int_{\Omega \cap \mathbb{R}_{-,x_0}^N} |\nabla v_j(x)|^2 dx \right] \\ &\leq \frac{1}{S_{\alpha,\beta}} \left[ \alpha \int_{\Omega} |\nabla v_j(x)|^2 dx + \beta \int_{\Omega} |\nabla v_j(x)|^2 dx \right] \\ \|v_j\|_{L^{2^*}}^2 &\leq \frac{1}{S_{\alpha,\beta}} \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx. \end{aligned} \quad (17)$$

where  $\mathbb{R}_{+,x_0}^N = \{x = (x', x_N) \in \mathbb{R}^N, \ x' \in \mathbb{R}^{N-1}, \ x_N > x_{0N}\}$  and  $\mathbb{R}_{-,x_0}^N = \{x = (x', x_N) \in \mathbb{R}^N, \ x' \in \mathbb{R}^{N-1}, \ x_N < x_{0N}\}$  with  $x_{0N}$  is such that  $x_0 = (x', x_{0N})$ .

We claim that  $u \not\equiv 0$ .

Indeed, suppose that  $u \equiv 0$ . From (14) we obtain

$$\int_{\Omega} p(x) |\nabla v_j|^2 dx = S(p) + o(1),$$

then

$$\lim_{j \rightarrow +\infty} \int_{\Omega} p(x) |\nabla v_j|^2 dx = S(p).$$

From (15) we see that

$$\lim_{j \rightarrow +\infty} \|v_j\|_{L^{2^*}} = 1.$$

Or (17) gives that

$$\|v_j\|_{L^{2^*}}^2 S_{\alpha,\beta} \leq \int_{\Omega} p(x) |\nabla v_j|^2 dx.$$

Passing to limit in the previous inequality we obtain  $S_{\alpha,\beta} \leq S(p)$ . This contradicts the hypothesis  $S(p) < S_{\alpha,\beta}$ . Consequently  $u \not\equiv 0$ .

Now, we deduce from (16) and (17) that

$$S(p) \leq S(p) \|u\|_{L^{2^*}}^{2^*} + \frac{S(p)}{S_{\alpha,\beta}} \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx + o(1). \quad (18)$$

Combining (14) and (18) we obtain

$$\int_{\Omega} p(x)|\nabla u(x)|^2 + \int_{\Omega} p(x)|\nabla v_j(x)|^2 dx \leq S(p)\|u\|_{L^{2^*}}^2 + \frac{S(p)}{S_{\alpha,\beta}} \int_{\Omega} p(x)|\nabla v_j(x)|^2 dx + o(1).$$

Thus

$$\int_{\Omega} p(x)|\nabla u(x)|^2 dx \leq S(p)\|u\|_{L^{2^*}}^2 + \left[ \frac{S(p)}{S_{\alpha,\beta}} - 1 \right] \int_{\Omega} p(x)|\nabla v_j(x)|^2 dx + o(1).$$

Since  $S(p) < S_{\alpha,\beta}$ , we deduce

$$\int_{\Omega} p(x)|\nabla u(x)|^2 dx \leq S(p)\|u\|_{L^{2^*}}^2, \quad (19)$$

Therefore

$$\int_{\Omega} p(x)|\nabla u(x)|^2 dx = S(p)\|u\|_{L^{2^*}}^2.$$

this means that  $u$  is a minimum of  $S(p)$ .  $\square$

**Lemma 2** Assume that there exists  $x_0$  in the interior of  $\Gamma_1$  such that the (g.c.) holds. Then

$$S(p) < S_{\alpha,\beta}.$$

**Proof.** Let  $\{\lambda_i(x_0)\}_{1 \leq i \leq N-1}$ , denote the principal curvatures and  $H(x_0) = \frac{1}{N-1} \sum_{i=1}^{N-1} \lambda_i(x_0)$

the mean curvature at  $x_0$  with respect to the unit normal.

For the simplicity, we suppose that  $x_0 = 0$ . Therefore we note  $\{\lambda_i\}_{1 \leq i \leq N-1}$  the principal

curvatures at 0 and  $H(0) = \frac{1}{N-1} \sum_{i=1}^{N-1} \lambda_i$ . Let  $R > 0$ , such that

$$B(R) \cap \Omega_1 = \{(x', x_N) \in B(R); x_N > \rho(x')\}$$

$$B(R) \cap \Omega_2 = \{(x', x_N) \in B(R); x_N < \rho(x')\}$$

$$B(R) \cap \Gamma_1 = \{(x', x_N) \in B(R); x_N = \rho(x')\}$$

where  $x' = (x_1, x_2, \dots, x_{N-1})$  and  $\rho(x')$  is defined by

$$\rho(x') = \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3).$$

We note that the condition (g.c.) implies that  $\rho(x') \geq 0$ .

Let us define, for  $\varepsilon > 0$  and for  $t \in ]0, 1[$  the function

$$u_{0,\varepsilon,t}(x) = \begin{cases} \frac{\varphi(x)}{(\varepsilon + |\mathbf{x}'|^2 + t^{-\frac{N-2}{2}} \mathbf{x}_N^2)^{\frac{N-2}{2}}} & \text{If } x_N > 0 \\ \frac{\varphi(x)}{(\varepsilon + |\mathbf{x}'|^2 + (1-t)^{-\frac{N-2}{2}} \mathbf{x}_N^2)^{\frac{N-2}{2}}} & \text{If } x_N < 0 \end{cases}$$



where  $\varphi$  is a radial  $C^\infty$ -function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{R}{4} \\ 0 & \text{if } |x| \geq \frac{R}{4}. \end{cases}$$

There exists  $t_0 = \frac{(\frac{\alpha}{\beta})^{\frac{N}{2}}}{1 + (\frac{\alpha}{\beta})^{\frac{N}{2}}}$  such that

$$\inf_{t \in [0, 1]} \frac{(\alpha t^{\frac{2}{2^*}} + \beta (1-t)^{\frac{2}{2^*}}) S}{2^{\frac{2}{N}}} = \frac{(\alpha t_0^{\frac{2}{2^*}} + \beta (1-t_0)^{\frac{2}{2^*}}) S}{2^{\frac{2}{N}}} = \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S.$$

We note  $u_{0,\varepsilon}(x) = u_{0,\varepsilon,t_0}(x)$ . Set, for  $i \in \{1, 2\}$ ,

$$Q_i(u) = \frac{\int_{\Omega_i} |\nabla u|^2 dx}{\left( \int_{\Omega} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}$$

and

$$Q(u) = \alpha Q_1 + \beta Q_2(u).$$

In order to obtain the result of Lemma 2, we use  $u_{0,\varepsilon}$  as a test function for  $S(p)$ .

From ([2], page 13), direct computation gives

$$Q_1(u_{0,\varepsilon}) = \begin{cases} \frac{t_0^{\frac{2}{2^*}} S}{2^{\frac{2}{N}}} + S H(0) A(N) \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N = 3 \\ \frac{t_0^{\frac{2}{2^*}} S}{2^{\frac{2}{N}}} + S H(0) A(N) \varepsilon^{\frac{1}{2}} + O(\varepsilon |\ln(\varepsilon)|) & \text{if } N \geq 4 \end{cases} \quad (20)$$

and

$$Q_2(u_{0,\varepsilon}) = \begin{cases} \frac{(1-t_0)^{\frac{2}{2^*}} S}{2^{\frac{2}{N}}} - S H(0) A(N) \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N = 3 \\ \frac{(1-t_0)^{\frac{2}{2^*}} S}{2^{\frac{2}{N}}} - S H(0) A(N) \varepsilon^{\frac{1}{2}} + O(\varepsilon |\ln(\varepsilon)|) & \text{if } N \geq 4 \end{cases} \quad (21)$$

where  $A(N)$  is a positive constant.

Combining (20) and (21) we see that,

$$Q(u_{0,\varepsilon}) = \begin{cases} \frac{(\alpha t_0^{\frac{2}{2^*}} + \beta (1-t_0)^{\frac{2}{2^*}}) S}{2^{\frac{2}{N}}} - (\beta - \alpha) S H(0) A(N) \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N = 3 \\ \frac{(\alpha t_0^{\frac{2}{2^*}} + \beta (1-t_0)^{\frac{2}{2^*}}) S}{2^{\frac{2}{N}}} - (\beta - \alpha) S H(0) A(N) \varepsilon^{\frac{1}{2}} + O(\varepsilon |\ln(\varepsilon)|) & \text{if } N \geq 4. \end{cases}$$

Therefore, using the definition of  $t_0$ , we obtain

$$Q(u_{0,\varepsilon}) \leq \begin{cases} \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S - (\beta - \alpha) S H(0) A(N) \varepsilon^{\frac{1}{2}} |\ln(\varepsilon)| + O(\varepsilon^{\frac{1}{2}}) & \text{if } N = 3 \\ \left( \frac{\alpha^{\frac{N}{2}} + \beta^{\frac{N}{2}}}{2} \right)^{\frac{2}{N}} S - (\beta - \alpha) S H(0) A(N) \varepsilon^{\frac{1}{2}} + O(\varepsilon |\ln(\varepsilon)|) & \text{if } N \geq 4. \end{cases}$$

Finally, Since  $\beta > \alpha$  and  $H(0) > 0$  then we obtain the desired result.  $\square$

**Remark 2** (see [2]): By looking at the previous proof, it follows that we can relax the condition (g.c.) by allowing some of the  $\lambda_i$ 's to be negative with mean curvature positive.

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